Forming an LP Dual

Burr, we shouldn't be fighting. You misunderstood me when I said we needed a dual.

Hamilton, you're going to die.

I heard that today's lecture is about forming the dual.

Tim, the turkey

Aaron Burr and Alexander Hamilton (not really)

Forming Dual LPs in General

The first thing to remember is that every linear program has a dual, and there are simple rules for formulating the dual.

Some of the rules are common to all linear programs. These are the first rules to learn.

- Every linear program has a unique dual.
- The dual of the dual linear program is the original linear program.
- We will explain rules for taking the dual of a maximization problem.

FIRST RULES

1. There is a dual variable for every constraint
2. The constraint matrix of the dual is the transpose of the constraint matrix of the primal.
3. The objective function and RHS are swapped.

Ollie

A "standard" primal-dual pair

The Primal LP

Maximize $z = 1x_1 + 2x_2 + 3x_3 + 4x_4$

subject to

$$8x_1 + 9x_2 + 10x_3 + 11x_4 \le 5 \qquad y_1$$
$$12x_1 + 13x_2 + 14x_3 + 15x_4 \le 6 \qquad y_2$$
$$16x_1 + 17x_2 + 18x_3 + 19x_4 \le 7 \qquad y_3$$
$$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0,\ x_4 \ge 0$$

Dual Variable: y_1, y_2, y_3

The Dual LP

Minimize $v = 5y_1 + 6y_2 + 7y_3$

subject to

$$8y_1 + 12y_2 + 16y_3 \ge 1$$
$$9y_1 + 13y_2 + 17y_3 \ge 2$$
$$10y_1 + 14y_2 + 18y_3 \ge 3$$
$$11y_1 + 15y_2 + 19y_3 \ge 4$$
$$y_1 \ge 0\ y_2 \ge 0\ y_3 \ge 0$$

What makes this pair of LPs the "standard" one?

All linear programs have an objective, linear inequalities and inequalities, and variables.

In the "standard" primal-dual pair (not to be confused with an LP in standard form), the max problem has "≤ constraints" and all variables are non-negative. The dual is a minimization problem with "≥ constraints" and non-negativity constraints. It's important to remember this, as we shall soon see.

Ollie

We will show how to form dual linear programs using the SOB approach.

SOB approach! I object! We need to optimize family values, and this isn't the way to go.

Don't worry Reverend Falwell. As usual, your fears as absolutely groundless.

Nooz, the most trusted name in fox.

Reverend Jerry Falwell

Cleaver

Hi, I'm Art Benjamin. I invented the SOB approach. The letters stand for "sensible, odd, and bizarre".

Professor Orlin used to use "standard" but instead of "sensible" which is OK.

For the max problem:

Primal	max	
i-th constraint	≤	S
i-th constraint	=	O
i-th constraint	≥	B

S: sensible

O: odd

B: bizarre

In the max problem, we expect the constraints to be ≤. This is ***sensible***. But if we get an "= constraint" we are only a little surprised. We find this ***odd***. However, a "≥ constraint" is ***bizarre***.

I also do mathemagic, the art of mental calculation. You can check out my website at http://www.math.hmc.edu/~benjamin/mathemagics.htm

For the min problem:

Dual	min	
i-th constraint	≥	S
i-th constraint	=	O
i-th constraint	≤	B

S: sensible

O: odd

B: bizarre

In the min problem, we expect the constraints to be ≥. This is *sensible*. But if we get an "= constraint" we are only a little surprised. We find this *odd*. However, a "≤ constraint" is *bizarre*.

Constraints on variables for all LPS

I found that students really had trouble remembering the rules for forming the LP dual until I came up with the SOB approach.

i-th variable	≥ 0	S
i-th variable	free	O
i-th variable	≤ 0	B

For all linear programs, we expect variables to be nonnegative. This is ***sensible***. But occasionally we get variables that are unconstrained in sign. This is ***odd***. If we ever get variables to that are required to be non-positive, we view it as ***bizarre***.

Combining all of the rules

Primal max	Dual min	
i-th constraint ≤	i-th variable ≥ 0	S
i-th constraint =	i-th variable free	O
i-th constraint ≥	i-th variable ≤ 0	B
i-th variable ≥ 0	i-th constraint ≥	S
i-th variable free	i-th constraint =	O
i-th variable ≤ 0	i-th constraint ≤	B

If the i-th constraint in the primal maximization problem is a "≥ constraint", then the i-th variable in the dual problem is ≤ 0.
In this case, both constraints are bizarre.

If the i-th variable in the problem maximization problem is ≤ 0, then the i-th constraint in the dual problem is an "≤ constraint".
In this case, both constraints are bizarre.

Rules for creating a dual linear program

The Primal LP

		Dual Variable
Maximize	$z = 1x_1 + 2x_2 + 3x_3 + 4x_4$	
subject to	$8x_1 + 9x_2 + 10x_3 + 11x_4 \;?\; 5$	y_1
	$12x_1 + 13x_2 + 14x_3 + 15x_4 \;?\; 6$	y_2
	$16x_1 + 17x_2 + 18x_3 + 19x_4 \;?\; 7$	y_3
	$x_1 \;?\quad x_2 \;?\quad x_3 \;?\quad x_4 \;?$	

The Dual LP

Minimize $v = 5y_1 + 6y_2 + 7y_3$

subject to

$$8y_1 + 12y_2 + 16y_3 \;?\; 1$$
$$9y_1 + 13y_2 + 17y_3 \;?\; 2$$
$$10y_1 + 14y_2 + 18y_3 \;?\; 3$$
$$11y_1 + 15y_2 + 19y_3 \;?\; 4$$
$$y_1 \;?\quad y_2 \;?\quad y_3 \;?$$

The SOB approach
sensible, odd, and bizarre

The Primal LP

$$\begin{array}{llll}
\text{Max} & z = 1x_1 + 2x_2 + 3x_3 + 4x_4 & & \\
\text{s.t.} & 8x_1 + 9x_2 + 10x_3 + 11x_4 & \geq & 5 \quad \text{B} \\
 & 12x_1 + 13x_2 + 14x_3 + 15x_4 & = & 6 \quad \text{O} \\
 & 16x_1 + 17x_2 + 18x_3 + 19x_4 & \leq & 7 \quad \text{S} \\
 & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \text{ uis}, \quad x_4 \leq 0\ ? & &
\end{array}$$

S S O B

uis. unconstrained in sign, also known as free.

This is a chance to practice determining which are constraints are S, O, or B. Then try to formulate the dual before turning to the next page by using the rules on page 9.

The SOB approach
sensible, odd, and bizarre

The Dual LP

$$
\begin{array}{llll}
\text{Min} & v = 5y_1 + 6y_2 + 7y_3 & & \\
\text{s.t} & 8y_1 + 12y_2 + 16y_3 & \geq 1 & \text{S} \\
 & 9y_1 + 13y_2 + 17y_3 & \geq 2 & \text{S} \\
 & 10y_1 + 14y_2 + 18y_3 & = 3 & \text{O} \\
 & 11y_1 + 15y_2 + 19y_3 & \leq 4 & \text{B} \\
 & y_1 \leq 0, \quad y_2 \text{ uis} \quad y_3 \geq 0 & &
\end{array}
$$

B O S

Tim, you have asked a very deep question. We'll give a partial answer next.

Recall that the optimal dual variables are also the shadow prices. We will show that the rules for forming the signs of the dual variables correspond to what we know about shadow prices.

This whole approach seems a bit bizarre. Why do these rules work at all?

Ollie

The Primal LP

		Dual Variable
Max	$z = 1x_1 + 2x_2 + 3x_3 + 4x_4$	
s.t.	$8x_1 + 9x_2 + 10x_3 + 11x_4 \geq 5$	y_1
	$12x_1 + 13x_2 + 14x_3 + 15x_4 = 6$	y_2
	$16x_1 + 17x_2 + 18x_3 + 19x_4 \leq 7$	y_3
	$x_1 \geq 0$, $x_2 \geq 0$, x_3 uis, $x_4 \leq 0$?	

The first constraint is a "≤ constraint". If we increase the RHS from 5 to 5+Δ, then the problem becomes more constrained, and there are fewer options. The optimal objective value either stays the same or goes down. So the shadow price $y_1 \leq 0$. A similar argument shows that $y_3 \geq 0$. As for modifying the RHS of constraint 2. Changing the 6 to 6 + Δ will change the feasible region, but we don't know whether the opt objective will go up or down. And so y_2 is free.

Ollie

Last Slide

And on that note, we end our tutorial on on forming LP duals. We hope that this tutorial has been informative, and we look forward to seeing you again at the next tutorial..

Cleaver