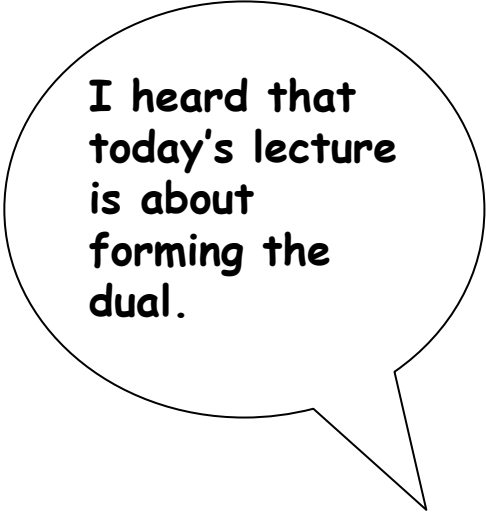
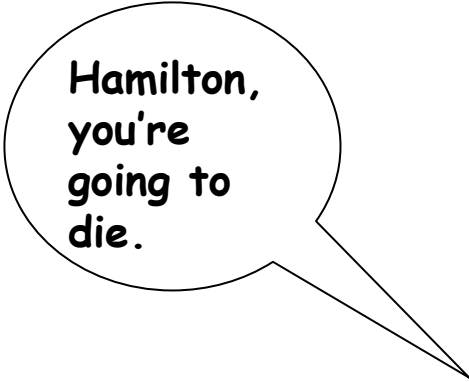



Forming an LP Dual



I heard that today's lecture is about forming the dual.



Hamilton, you're going to die.



Burr, we shouldn't be fighting. You misunderstood me when I said we needed a dual.

Tim, the turkey

Aaron Burr and Alexander
Hamilton (not really)

Forming Dual LPs in General

The first thing to remember is that every linear program has a dual, and there are simple rules for formulating the dual.

Some of the rules are common to all linear programs. These are the first rules to learn.

- Every linear program has a unique dual.
- The dual of the dual linear program is the original linear program.
- We will explain rules for taking the dual of a maximization problem.

FIRST RULES

1. There is a dual variable for every constraint
2. The constraint matrix of the dual is the transpose of the constraint matrix of the primal.
3. The objective function and RHS are swapped.

A “standard” primal-dual pair

The Primal LP

$$\begin{aligned} \text{Maximize } z = & 1 x_1 + 2 x_2 + 3 x_3 + 4 x_4 \\ \text{subject to } & 8 x_1 + 9 x_2 + 10 x_3 + 11 x_4 \leq 5 \\ & 12 x_1 + 13 x_2 + 14 x_3 + 15 x_4 \leq 6 \\ & 16 x_1 + 17 x_2 + 18 x_3 + 19 x_4 \leq 7 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \geq 0 \end{aligned}$$

Dual
Variable

y_1

y_2

y_3

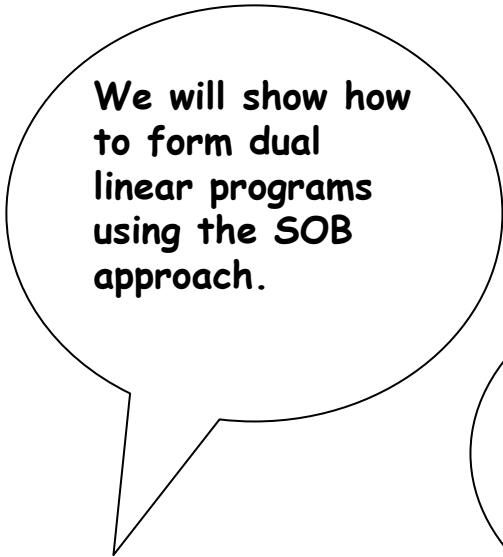
The Dual LP

$$\begin{aligned} \text{Minimize } v = & 5 y_1 + 6 y_2 + 7 y_3 \\ \text{subject to } & 8 y_1 + 12 y_2 + 16 y_3 \geq 1 \\ & 9 y_1 + 13 y_2 + 17 y_3 \geq 2 \\ & 10 y_1 + 14 y_2 + 18 y_3 \geq 3 \\ & 11 y_1 + 15 y_2 + 19 y_3 \geq 4 \\ & y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0 \end{aligned}$$

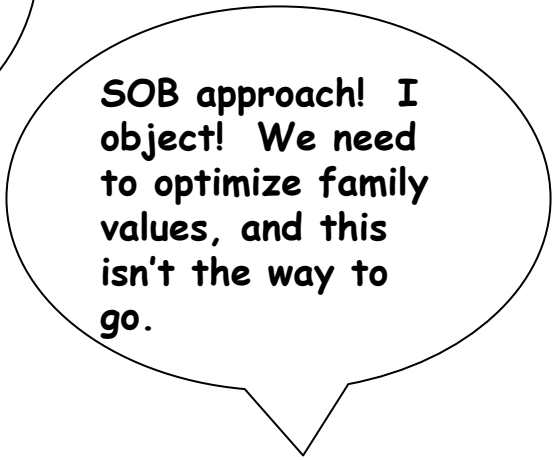
What makes
this pair of
LPs the
"standard"
one?

All linear programs have an objective,
linear inequalities and equalities, and
variables.

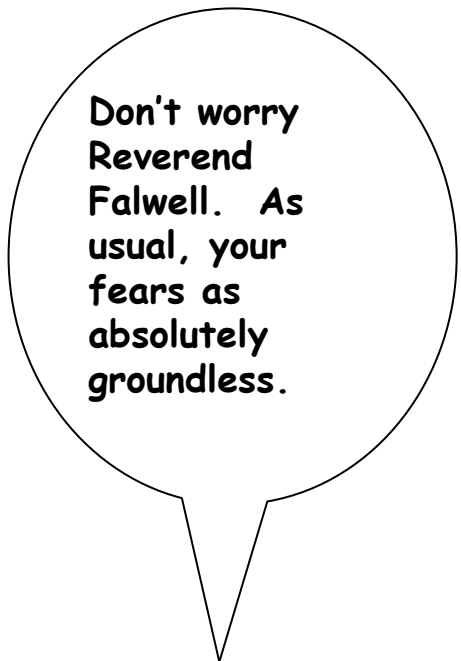
In the "standard" primal-dual pair
(not to be confused with an LP in
standard form), the max problem has
" \leq constraints" and all variables are
non-negative. The dual is a
minimization problem with " \geq
constraints" and non-negativity
constraints. It's important to
remember this, as we shall soon see.



We will show how to form dual linear programs using the SOB approach.



SOB approach! I object! We need to optimize family values, and this isn't the way to go.



Don't worry Reverend Falwell. As usual, your fears are absolutely groundless.

Nooz, the most trusted name in fox.

Reverend Jerry Falwell

Cleaver

Hi, I'm Art Benjamin. I invented the SOB approach. The letters stand for "sensible, odd, and bizarre".

Professor Orlin used to use "standard" but instead of "sensible" which is OK.

For the max problem:

Primal	max	
i-th constraint	\leq	S
i-th constraint	=	O
i-th constraint	\geq	B

S: sensible

O: odd

B: bizarre

In the max problem, we expect the constraints to be \leq . This is sensible. But if we get an "=" constraint" we are only a little surprised. We find this odd. However, a " \geq constraint" is bizarre.

I also do
 mathemagic, the
 art of mental
 calculation. You
 can check out my
 website at
<http://www.math.hmc.edu/~benjamin/mathemagics.htm>

For the min problem:

Dual	min	
i-th constraint	\geq	S
i-th constraint	$=$	O
i-th constraint	\leq	B

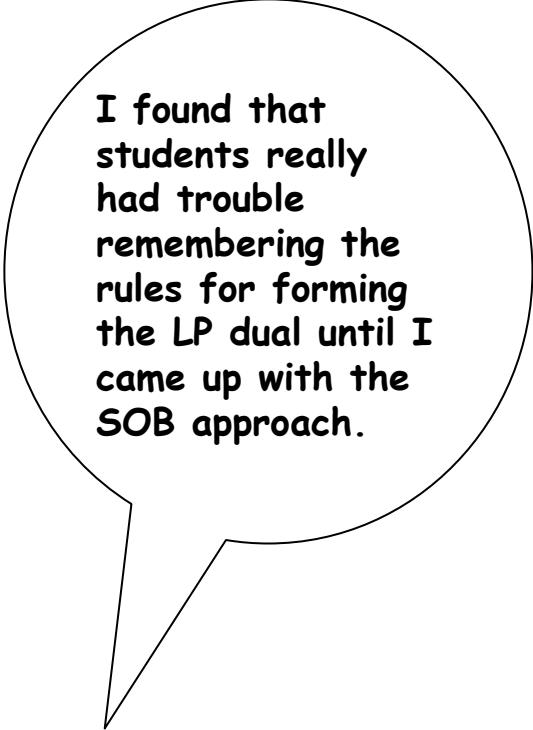
S: sensible

O: odd

B: bizarre

In the min problem, we expect the constraints to be \geq . This is **sensible**. But if we get an “= constraint” we are only a little surprised. We find this **odd**. However, a “ \leq constraint” is **bizarre**.

Constraints on variables for all LPS



I found that students really had trouble remembering the rules for forming the LP dual until I came up with the SOB approach.

i-th variable	≥ 0	S
i-th variable	free	O
i-th variable	≤ 0	B

For all linear programs, we expect variables to be nonnegative. This is sensible. But occasionally we get variables that are unconstrained in sign. This is odd. If we ever get variables that are required to be non-positive, we view it as bizarre.

Combining all of the rules

Primal	max	Dual	min	
i-th constraint	\leq	i-th variable	≥ 0	S
i-th constraint	$=$	i-th variable	free	O
i-th constraint	\geq	i-th variable	≤ 0	B
i-th variable	≥ 0	i-th constraint	\geq	S
i-th variable	free	i-th constraint	$=$	O
i-th variable	≤ 0	i-th constraint	\leq	B

If the i-th constraint in the primal maximization problem is a “ \geq constraint”, then the i-th variable in the dual problem is ≤ 0 .

In this case, both constraints are bizarre.

If the i-th variable in the problem maximization problem is ≤ 0 , then the i-th constraint in the dual problem is an “ \leq constraint”.

In this case, both constraints are bizarre.

Rules for creating a dual linear program

The Primal LP

$$\begin{array}{l}
 \text{Maximize } z = 1x_1 + 2x_2 + 3x_3 + 4x_4 \\
 \text{subject to} \\
 \quad 8x_1 + 9x_2 + 10x_3 + 11x_4 \quad ? \quad 5 \\
 \quad 12x_1 + 13x_2 + 14x_3 + 15x_4 \quad ? \quad 6 \\
 \quad 16x_1 + 17x_2 + 18x_3 + 19x_4 \quad ? \quad 7 \\
 \quad x_1 \quad ? \quad x_2 \quad ? \quad x_3 \quad ? \quad x_4 \quad ?
 \end{array}$$

Dual
Variable

y_1

y_2

y_3

The Dual LP

$$\begin{array}{l}
 \text{Minimize } v = 5y_1 + 6y_2 + 7y_3 \\
 \text{subject to} \\
 \quad 8y_1 + 12y_2 + 16y_3 \quad ? \quad 1 \\
 \quad 9y_1 + 13y_2 + 17y_3 \quad ? \quad 2 \\
 \quad 10y_1 + 14y_2 + 18y_3 \quad ? \quad 3 \\
 \quad 11y_1 + 15y_2 + 19y_3 \quad ? \quad 4 \\
 \quad y_1 \quad ? \quad y_2 \quad ? \quad y_3 \quad ?
 \end{array}$$

The SOB approach sensible, odd, and bizarre

The Primal LP

$$\begin{aligned} \text{Max } z = & 1 x_1 + 2 x_2 + 3 x_3 + 4 x_4 \\ \text{s.t. } & 8 x_1 + 9 x_2 + 10 x_3 + 11 x_4 \geq 5 \\ & 12 x_1 + 13 x_2 + 14 x_3 + 15 x_4 = 6 \\ & 16 x_1 + 17 x_2 + 18 x_3 + 19 x_4 \leq 7 \\ & x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \text{ uis}, \quad x_4 \leq 0 ? \end{aligned}$$

B

O

S

S

S

O

B

uis. unconstrained in sign, also known as free.

This is a chance to practice determining which are constraints are S, O, or B. Then try to formulate the dual before turning to the next page by using the rules on page 9.

The SOB approach sensible, odd, and bizarre

The Dual LP

$$\text{Min } v = 5 y_1 + 6 y_2 + 7 y_3$$

$$\text{s.t } 8 y_1 + 12 y_2 + 16 y_3 \geq 1 \quad \boxed{\text{S}}$$

$$9 y_1 + 13 y_2 + 17 y_3 \geq 2 \quad \boxed{\text{S}}$$

$$10 y_1 + 14 y_2 + 18 y_3 = 3 \quad \boxed{\text{O}}$$

$$11 y_1 + 15 y_2 + 19 y_3 \leq 4 \quad \boxed{\text{B}}$$

$$y_1 \leq 0, \quad y_2 \text{ uis } y_3 \geq 0$$

B

O

S

This whole approach seems a bit bizarre. Why do these rules work at all?

Tim, you have asked a very deep question. We'll give a partial answer next.

Recall that the optimal dual variables are also the shadow prices. We will show that the rules for forming the signs of the dual variables correspond to what we know about shadow prices.

The Primal LP

$$\begin{aligned} \text{Max } z = & 1 x_1 + 2 x_2 + 3 x_3 + 4 x_4 \\ \text{s.t.} & 8 x_1 + 9 x_2 + 10 x_3 + 11 x_4 \geq 5 \\ & 12 x_1 + 13 x_2 + 14 x_3 + 15 x_4 = 6 \\ & 16 x_1 + 17 x_2 + 18 x_3 + 19 x_4 \leq 7 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \text{ free}, x_4 \leq 0 \end{aligned}$$

Dual
Variable


y_1

y_2

y_3

The first constraint is a " \leq constraint". If we increase the RHS from 5 to $5+\Delta$, then the problem becomes more constrained, and there are fewer options. The optimal objective value either stays the same or goes down. So the shadow price $y_1 \leq 0$. A similar argument shows that $y_3 \geq 0$. As for modifying the RHS of constraint 2. Changing the 6 to $6 + \Delta$ will change the feasible region, but we don't know whether the opt objective will go up or down. And so y_2 is free.

Last Slide



And on that note, we end our tutorial on forming LP duals. We hope that this tutorial has been informative, and we look forward to seeing you again at the next tutorial..

Cleaver